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Hereditariness, Strongness and Relationship between Brown-McCoy and Behrens Radicals

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Abstract. In this paper we explore the properties of being hereditary and being strong among the radicals of associative rings, and prove certain results such as a relationship between Brown-McCoy and Behrens radicals.

MSC 2000: 16N80

I.

In this paper rings are all associative, but not necessarily with a unit element. As usual, $I ◁ A$ and $L ◁ l A$ ($R ◁ r A$) denote that $I$ is an ideal and $L$ is a left ideal ($R$ is a right ideal) in $A$, respectively. $A^o$ will stand for the ring on the additive group $(A, +)$ with multiplication $xy = 0$, for all $x, y ∈ A$.

Let us recall that a (Kurosh-Amitsur) radical $γ$ is a class of rings which is closed under homomorphisms, extensions ($I$ and $A/I$ in $γ$ imply $A$ in $γ$), and has the inductive property (if $I_1 ⊆ ⋯ ⊆ I_λ ⊆ ⋯$ is a chain of ideals, $A = ∪I_λ$, and each $I_λ$ is in $γ$, then $A$ is in $γ$).

The first author carried out research within the framework of the Hungarian-Mongolian cultural exchange program at the A. Rényi Institute of Mathematics HAS, Budapest. He gratefully acknowledges the kind hospitality and also the support of OTKA Grant # T29525.

0138-4821/93 $ 2.50 © 2001 Heldermann Verlag
The unique largest $\gamma$-ideal $\gamma(A)$ of $A$ is then the $\gamma$-radical of $A$. A hereditary radical containing all nilpotent rings is called a supernilpotent radical. Let $\mathcal{M}$ be a class of rings. Put

$$\overline{\mathcal{M}} = \{ A \mid \text{every ideal of } A \text{ is in } \mathcal{M} \}.$$ 

A radical $\gamma$ is said to be principally left (right) hereditary if $a \in A \in \gamma$ implies $Aa \in \gamma$ ($aA \in \gamma$, respectively). A radical $\gamma$ is said to be left (right) strong if $L \triangleleft l A$ ($R \triangleleft r A$) and $L \in \gamma$ ($R \in \gamma(A)$, respectively). A radical $\gamma$ is normal if $\gamma$ is left strong and principally left hereditary. We shall make use of the following condition a left ideal $L$ of a ring $A$ may satisfy with respect to a class $\mathcal{M}$ of rings:

$$(*) \quad L \triangleleft l A \text{ and } Lz \in \mathcal{M} \text{ for all } z \in L \cup \{1\}.$$ 

A radical $\gamma$ is said to be principally left strong if $L \subseteq \gamma(A)$ whenever the left ideal $L$ of a ring $A$ satisfies condition $(*$) with respect to the class $\gamma(= \mathcal{M})$. Principally right strongness is defined analogously.

We will focus on two conditions that a class $\mathcal{M}$ can satisfy.

(H) If $A^o \in \mathcal{M}$ then $S \in \mathcal{M}$ for every subring $S \subseteq A^o$.

(Z) If $A \in \mathcal{M}$ then $A^o \in \mathcal{M}$.

A class $\mathcal{M}$ of rings is said to be regular if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the upper radical operator $U$ yields a radical class

$$UM = \{ A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M} \}.$$ 

Recall that the Baer radical $\beta$ is the upper radical determined by all prime rings, the Brown-McCoy radical $G$ is the upper radical determined by all simple rings with unity element, and the Behrens radical $B$ is the upper radical of all subdirectly irreducible rings having a nonzero idempotent in their hearts.

The lower principally left strong radical construction $L_{ps}(\mathcal{M})$ is similar to the lower (left) strong radical construction $L_s(\mathcal{M})$ (see [1]).

We shall construct the lower principally left strong radical (see also [7]) in the following way. Let $\mathcal{M}$ be a homomorphically closed class of rings and define $\mathcal{M} = \mathcal{M}_1$,

$$\mathcal{M}_{\alpha+1} = \left\{ A \mid \text{every nonzero homomorphic image of } A \text{ has a nonzero left ideal with (*) in } \mathcal{M}_\alpha \text{ or a nonzero ideal } I \in \mathcal{M}_\alpha \right\}$$

for ordinals $\alpha \geq 1$ and $\mathcal{M}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{M}_\alpha$ for limit ordinals $\lambda$. In particular,

$$\mathcal{M}_2 = \left\{ A \mid \text{every nonzero homomorphic image of } A \text{ has a nonzero left ideal with (*) in } \mathcal{M} \text{ or a nonzero ideal } I \in \mathcal{M} \right\}.$$ 

The class $L_{ps}(\mathcal{M}) = \bigcup_\alpha \mathcal{M}_\alpha$ is called the lower principally left strong radical class. As shown in [6] $L_{ps}(\mathcal{M})$ is the smallest principally left strong radical containing $\mathcal{M}$ and

$$\mathcal{M} \subseteq L(\mathcal{M}) \subseteq L_{ps}(\mathcal{M}) \subseteq L_s(\mathcal{M}).$$
For any class $\mathcal{M}$ let us define $\mathcal{M}^o = \{A \mid A^o \in \mathcal{M}\}$. It is easy to see that if $\mathcal{M}$ is a radical then so is $\mathcal{M}^o$. Let
\[ \gamma_l = \{A \in \gamma \mid \text{every left ideal of } A \text{ is in } \gamma\} \]
and
\[ \gamma_r = \{A \in \gamma \mid \text{every right ideal of } A \text{ is in } \gamma\}. \]

Next, we recall some results which will be used later on.

**Proposition 1.** [2, Lemma 1] Let $\gamma$ be a radical. If $S$ is a subring of a ring $A$ such that $S^o \in \gamma$, then also $(S^o)^o \in \gamma$ where $S^o$ denotes the ideal of $A$ generated by $S$.

**Proposition 2.** [5, Lemma 2.4] Let $\gamma$ be a radical. If $\beta(A)^o \in \gamma$, then $\beta(A) \in \gamma$.

**Proposition 3.** [2, Corollary 1] If $\mathcal{M} \subseteq \mathcal{M}^o$ then $\mathcal{L}(\mathcal{M}) \subseteq (\mathcal{L}(\mathcal{M}))^o$ and $\mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^o$.

**Proposition 4.** [4, Theorem 4] If a radical $\gamma$ is left strong and principally left hereditary, then $\gamma$ is normal.

**Proposition 5.** [2, Lemma 2] For any element $a$ of a ring $A$, $I = r(a)a$, where $r(a) = \{x \in A \mid ax = 0\}$ is an ideal of $Aa$ and $I^2 = 0$. In addition $Aa/I$ is a homomorphic image of $aA$.

**Proposition 6.** [5, Corollary 4.2] A radical $\gamma$ is hereditary and normal if and only if $\gamma$ is principally left strong, principally left hereditary and satisfies condition (H).

**Proposition 7.** [7, Theorem 6] A radical $\gamma$ is normal if and only if $\gamma$ is principally left or right hereditary and principally left or right strong.

**Proposition 8.** [6, Theorem 3.3] Let $\mathcal{M}$ be a homomorphically closed class of rings satisfying:
1) $\mathcal{M}$ contains all zero rings;
2) $\mathcal{M}$ is hereditary;
3) if $I \triangleleft A$, $I^2 = 0$ and $A/I \in \mathcal{M}$ then $A \in \mathcal{M}$.
Then $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{M}_2$.

**Proposition 9.** [5, Theorem 5.1] The Behrens radical class $\mathcal{B}$ is the largest principally left hereditary subclass of the Brown-McCoy radical class $\mathcal{G}$, in fact
\[ \mathcal{B} = \mathcal{MG}, \]
where
\[ \mathcal{MG} = \{A \mid Aa \in \mathcal{G} \text{ for all } a \in A\}. \]

A ring $A$ is said to be (right) strongly prime if every non-zero ideal $I$ of $A$ contains a finite subset $F$ such that $r_A(F) = 0$, where $r_A(F) = \{x \in A \mid Fx = 0\}$.

The (right) strongly prime radical $S$ is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring $A$,
\[ S(A) = \cap\{I \triangleleft A \mid A/I \text{ is strongly prime}\}. \]

It is known that the radical $S$ is special: so, in particular, $S$ is hereditary and contains the prime radical $\beta$.

**Proposition 10.** [3, Corollary 1] The (right) strongly prime radical $S$ is right strong.
II. Proposition 11. Let \( \gamma \) be a principally left strong radical satisfying the conditions (H) and (Z). Then the largest hereditary subclass \( \overline{\gamma} \) of \( \gamma \) will be principally left strong.

Proof. Let \( L \trianglelefteq A \) be such that \( L \in \overline{\gamma} \) and \( Lz \in \overline{\gamma} \) for every \( z \in L \). Let \( L^* \) be the ideal in \( A \) generated by \( L \), \( L^* = L + LA \) and suppose \( I \trianglelefteq L^* \). Then \( IL \trianglelefteq L \), \( IL \trianglelefteq I \) and \( ILz \trianglelefteq Lz \in \overline{\gamma} \) for all \( z \in L \). Since \( \gamma \) satisfies condition (H), \( \overline{\gamma} \) is hereditary, and so \( ILz \in \overline{\gamma} \) for all \( z \in IL \). Since \( \gamma \) is principally left strong \( IL \subseteq \gamma(I) \). We have

\[
I(L^*)^2 = I(L + LA)L^* = (IL + ILA)L^* \subseteq ILL^* \subseteq \gamma(I)L^* \subseteq \gamma(I).
\]

So \( I^3 \subseteq I(L^*)^2 \subseteq \gamma(I) \) and therefore \( I/\gamma(I) \) is nilpotent, implying \( I/\gamma(I) \in \beta \). We claim that \( I^\circ \in \gamma \). Since \( L \in \overline{\gamma} \subseteq \gamma \), by (Z) we conclude that \( L^\circ \in \gamma \). Now Proposition 1 implies that \( (L^*)^\circ \in \gamma \) and so by (H) it follows \( I^\circ \in \gamma \). Hence \( (I/\gamma(I))^\circ \in \gamma \cap \beta \) and applying Proposition 2 and taking into consideration that \( I/\gamma(I) \) is nilpotent, we get

\[
I/\beta(I) = \beta(I/\gamma(A)) \subseteq \gamma.
\]

Thus \( I \in \gamma \) and so \( \overline{\gamma} \) is principally left strong.

Corollary 12. If a class \( \mathcal{M} \) is hereditary and satisfies (Z) then \( \mathcal{L}_{ps}(\mathcal{M}) \) is hereditary.

Proof. By Proposition 3, we have \( \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_{\gamma}(\mathcal{M}) \subseteq (\mathcal{L}_{\gamma}(\mathcal{M}))^\circ \). Let \( A \in \mathcal{L}_{ps}(\mathcal{M}) \) then we get \( A^\circ \in \mathcal{L}_{\gamma}(\mathcal{M}) \) and so \( A^\circ \in \mathcal{L}(\mathcal{M}) \). Since \( \mathcal{L}(\mathcal{M}) \) is hereditary, we conclude that \( A^\circ \in \mathcal{L}(\mathcal{M}) \) and so \( A^\circ \in \mathcal{L}_{ps}(\mathcal{M}) \). This means that \( \mathcal{L}_{ps}(\mathcal{M}) \) satisfies the conditions (Z) and (H). By Proposition 11, \( \mathcal{L}_{ps}(\mathcal{M}) \) is principally left strong and \( \mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_{ps}(\mathcal{M}) \) and this implies \( \mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_{ps}(\mathcal{M}) \).

Proposition 13. Let \( \gamma \) be a principally left strong radical satisfying the conditions (H) and (Z). Then \( \gamma_{\gamma} \) is left strong.

Proof. Let \( L \trianglelefteq A \) and \( L \in \gamma_{\gamma} \) and let \( K \) be a left ideal of \( L^* = L + LA \). Since \( L \in \gamma_{\gamma} \), \( kL \in \gamma \) for every \( k \in K \). Let \( R \trianglelefteq kL \). Then it is easy to see that \( RkL \in \gamma \), and by conditions (Z) and (H), \( R/kL \in \gamma \) and so \( R \in \gamma \). Hence \( kL \in \gamma_{\gamma} \) for every \( k \in K \). An argument similar to the proof of Proposition 5 will show that \( (Lk + r(k)k)/r(k)k \) is a homomorphic image of \( kL \), where \( r(k) = \{ x \in L^*/kx = 0 \} \). Hence \( (Lk + r(k)k)/r(k)k \in \gamma \). By (H) and (Z) we have \( r(k)k \in \gamma \) and so \( Lk \in \gamma \) for every \( k \in K \). Therefore \( Lk \subseteq \gamma(K) \) and \( LK \subseteq \gamma(K) \). Clearly

\[
K^3 \subseteq (L^*K)K \subseteq (LA^1K)K \subseteq LL^*K \subseteqLK \subseteq \gamma(K)
\]

hence \( K \in \gamma \) by Proposition 2.

The next result is a generalization of [2, Corollary 4].

Corollary 14. If \( \mathcal{M} \) is a right hereditary class with (Z), then \( \mathcal{L}_{ps}(\mathcal{M}) \) is one-sided hereditary and \( \mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_{\gamma}(\mathcal{M}) \) (i.e. \( \mathcal{L}_{ps}(\mathcal{M}) \) is left and right hereditary).
Proof. By Corollary 12, $\mathcal{L}_{ps}(\mathcal{M})$ satisfies condition (H). Let $A \in \mathcal{L}_{ps}(\mathcal{M})$. Then it is easy to see that $A^o \in \mathcal{L}_{ps}(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M})$ satisfies condition (Z). Hence $\mathcal{L}_{ps}(\mathcal{M})_r$ is a radical. By Proposition 13, $\mathcal{L}_{ps}(\mathcal{M})_r$ is left strong. Since $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r$ we get $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_{s}(\mathcal{M})$ and $\mathcal{L}_{ps}(\mathcal{M})_r = \mathcal{L}_{s}(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_{s}(\mathcal{M})$. Since $\mathcal{L}_{ps}(\mathcal{M})_r$ is right hereditary and left strong, we have that $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary.

Theorem 15. Let $\gamma \neq 0$ be a principally left strong radical with (Z) and (H). Then $\gamma_r$ is contained in $\gamma$ as a largest nonzero hereditary and normal subradical. Furthermore, $\gamma_l$ is contained in $\gamma$ as a largest non-zero hereditary principally left strong subradical.

Proof. Let $0 \neq A \in \gamma$. By (Z), $A^o \in \gamma$ and by (H), $A^o \in \gamma_r$. All zero-rings of $\gamma$ are in $\gamma_r$ and so $\gamma_r \neq 0$. Hence $\gamma_r$ satisfies conditions (Z) and (H). By Propositions 13, 6 and 4, $\gamma$ is normal and hereditary.

The second part of the theorem follows from Proposition 11.

Corollary 16. The largest left hereditary subclass $S_l$ of strongly prime radical $S$ is the largest normal radical contained in $S$.

Theorem 17. The following statements are equivalent for a radical $\gamma$.

1) $\gamma$ is hereditary and normal.

2) $\gamma$ is left or right principally hereditary, principally left or right strong and satisfies condition (H).

3) There exists a principally left (right, respectively) strong radical $\delta$ such that $\delta_r = \gamma$ ($\delta_l = \gamma$, respectively) and satisfies conditions (Z) and (H).

4) There exists a right (left, respectively) hereditary class $\mathcal{M}$ of rings satisfying (Z) such that $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}_{ps}^l(\mathcal{M})$, respectively), where $\mathcal{L}_{ps}^l(\mathcal{M})$ is principally right strong radical generated by $\mathcal{M}$.

Proof. 2) $\implies$ 1): By Proposition 7, $\gamma$ is normal and by Proposition 6, $\gamma$ is hereditary.

1) $\implies$ 3): We claim that $\gamma$ is one-sided hereditary. So let $L \lhd A \in \gamma$. Since $\gamma$ is normal, $\gamma$ is principally left hereditary, so $Aa \in \gamma$, for all $a \in L$. Therefore $Aa \cdot z \in \gamma$ for every $z \in Aa$. Hence $Aa \subseteq \gamma(L)$ for all $a \in L$, and this gives $L^2 \subseteq \gamma(L)$. Again, since $\gamma$ is normal and satisfies condition (Z), $A^o \in \gamma$ and by hereditariness $L^o \in \gamma$. Therefore $L \in \gamma$. Right hereditariness is proved analogously. Now we choose $\delta$ to be $\gamma$, $\delta = \gamma$ and we have $\gamma = \delta = \delta_l = \delta_r$.

3) $\implies$ 4): We choose $\mathcal{M} = \delta^o$ ($\mathcal{M} = \delta_l$, respectively). Then $\delta_r = \mathcal{L}_{ps}(\delta_l) = \mathcal{L}_{ps}(\mathcal{M})$ ($\delta_l = \mathcal{L}_{ps}^l(\delta_l) = \mathcal{L}_{ps}^l(\mathcal{M})$, respectively) by Proposition 13 and clearly $\delta_r$ satisfies (Z).

4) $\implies$ 2): By Corollary 14, $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}_{ps}^l(\mathcal{M})$) is one-sided hereditary and left strong. Hence by Proposition 4 it is normal. It is easy to see that $\gamma$ satisfies 2).

Proposition 18. Let $\gamma$ be a supernilpotent radical and let us assume that $\gamma_l = \gamma_r$ is the largest principally left hereditary subclass of $\gamma$ which we will denote by $\delta$. Then

$$\mathcal{L}_{ps}(\gamma) = \mathcal{L}_{ps}(\delta) \lor \gamma$$

where $\lor$ denotes the union in the lattice of all radicals (i.e. the lower radical determined by the union of the components).
Proof. Clearly $L_{ps}(\delta) \vee \gamma \subseteq L_{ps}(\gamma)$. Conversely, let $A \in L_{ps}(\gamma)$. Under our hypothesis, we can apply Proposition 8 and so $L_{ps}(\gamma) = \gamma^2$. Thus any non-zero homomorphic image $A'$ of $A$ has a non-zero $\gamma$-ideal or a non-zero left ideal $L$ such that $La \in \gamma$ for all $a \in L \cup \{1\}$. Using our hypothesis again, we conclude that $L \in \delta$ and therefore the $L_{ps}(\delta)$-radical of $A'$ is nonzero. Hence $A'$ has a nonzero ideal in $L_{ps}(\delta) \cup \gamma$ and so $A \in L_{ps}(\delta) \vee \gamma$. \qed

Corollary 19. $L_{ps}(G) = L_{ps}(B) \vee G$ and $G_2 = B_2 \vee G$.

Proof. By Proposition 9, the Brown-McCoy radical satisfies the assumption of Proposition 18, in fact, $MG = G_l = G_r = B$. \qed

Remark. This corollary can also be obtained as an application of Proposition 8 to the radicals $G$ and $B$.

Acknowledgement. The authors wish to express their indebtedness and gratitude to Prof. R. Wiegandt for his invaluable advice.

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Received May 11, 2000