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Hereditariness, Strongness and Relationship between Brown-McCoy and Behrens Radicals

S. Tumurbat H. Zand

Department of Algebra, University of Mongolia
P.O. Box 75, Ulaan Baatar 20, Mongolia
e-mail: tumur@www.com

Open University, Milton Keynes
MK7 6AA, England
e-mail: h.zand@open.ac.uk

Abstract. In this paper we explore the properties of being hereditary and being strong among the radicals of associative rings, and prove certain results such as a relationship between Brown-McCoy and Behrens radicals.

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I.

In this paper rings are all associative, but not necessarily with a unit element. As usual, $I \lhd A$ and $L \lhd_l A$ ($R \lhd_r A$) denote that $I$ is an ideal and $L$ is a left ideal ($R$ is a right ideal) in $A$, respectively. $A^o$ will stand for the ring on the additive group $(A, +)$ with multiplication $xy = 0$, for all $x, y \in A$.

Let us recall that a (Kurosh-Amitsur) radical $\gamma$ is a class of rings which is closed under homomorphisms, extensions ($I$ and $A/I$ in $\gamma$ imply $A$ in $\gamma$), and has the inductive property (if $I_1 \subseteq \cdots \subseteq I_\lambda \subseteq \cdots$ is a chain of ideals, $A = \cup I_\lambda$, and each $I_\lambda$ is in $\gamma$, then $A$ is in $\gamma$).

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The unique largest $\gamma$-ideal $\gamma(A)$ of $A$ is then the $\gamma$-radical of $A$. A hereditary radical containing all nilpotent rings is called a supernilpotent radical. Let $\mathcal{M}$ be a class of rings. Put
$$\overline{\mathcal{M}} = \{A \mid \text{every ideal of } A \text{ is in } \mathcal{M}\}.$$  

A radical $\gamma$ is said to be principally left (right) hereditary if $a \in A \in \gamma$ implies $Aa \in \gamma$ ($aA \in \gamma$, respectively). A radical $\gamma$ is said to be left (right) strong if $L \lhd L \in \gamma$ ($R \rhd R \in \gamma$) imply $L \subseteq \gamma(A)$ ($R \subseteq \gamma(A)$, respectively). A radical $\gamma$ is normal if $\gamma$ is left strong and principally left hereditary. We shall make use of the following condition a left ideal $L$ of a ring $A$ may satisfy with respect to a class $\mathcal{M}$ of rings:

($\ast$) $L \lhd L$ and $Lz \in \mathcal{M}$ for all $z \in L \cup \{1\}$.

A radical $\gamma$ is said to be principally left strong if $L \subseteq \gamma(A)$ whenever the left ideal $L$ of a ring $A$ satisfies condition ($\ast$) with respect to the class $\gamma(= \mathcal{M})$. Principally right strongness is defined analogously.

We will focus on two conditions that a class $\mathcal{M}$ can satisfy.

(H) If $A^o \in \mathcal{M}$ then $S \in \mathcal{M}$ for every subring $S \subseteq A^o$.

(Z) If $A \in \mathcal{M}$ then $A^o \in \mathcal{M}$.

A class $\mathcal{M}$ of rings is said to be regular if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the upper radical operator $\mathcal{U}$ yields a radical class

$$\mathcal{U}\mathcal{M} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}.$$  

Recall that the Baer radical $\beta$ is the upper radical determined by all prime rings, the Brown-McCoy radical $\mathcal{G}$ is the upper radical determined by all simple rings with unity element, and the Behrens radical $\mathcal{B}$ is the upper radical of all subdirectly irreducible rings having a nonzero idempotent in their hearts.

The lower principally left strong radical construction $\mathcal{L}_{ps}(\mathcal{M})$ is similar to the lower (left) strong radical construction $\mathcal{L}_s(\mathcal{M})$ (see [1]).

We shall construct the lower principally left strong radical (see also [7]) in the following way. Let $\mathcal{M}$ be a homomorphically closed class of rings and define $\mathcal{M} = \mathcal{M}_1$,

$$\mathcal{M}_{\alpha+1} = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (\ast) \text{ in } \mathcal{M}_\alpha \text{ or a nonzero} \\ \text{ideal } I \in \mathcal{M}_\alpha \end{array} \right\}$$

for ordinals $\alpha \geq 1$ and $\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$ for limit ordinals $\lambda$. In particular,

$$\mathcal{M}_2 = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (\ast) \text{ in } \mathcal{M} \text{ or a nonzero ideal} \\ I \in \mathcal{M} \end{array} \right\}.$$  

The class $\mathcal{L}_{ps}(\mathcal{M}) = \bigcup_{\alpha} \mathcal{M}_\alpha$ is called the lower principally left strong radical class. As shown in [6] $\mathcal{L}_{ps}(\mathcal{M})$ is the smallest principally left strong radical containing $\mathcal{M}$ and

$$\mathcal{M} \subseteq \mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}).$$
For any class $\mathcal{M}$ let us define $\mathcal{M}^o = \{ A \mid A^o \in \mathcal{M} \}$. It is easy to see that if $\mathcal{M}$ is a radical then so is $\mathcal{M}^o$. Let

$$\gamma = \{ A \in \gamma \mid \text{every left ideal of } A \text{ is in } \gamma \}$$

and

$$\gamma_r = \{ A \in \gamma \mid \text{every right ideal of } A \text{ is in } \gamma \}.$$

Next, we recall some results which will be used later on.

**Proposition 1.** [2, Lemma 1] Let $\gamma$ be a radical. If $S$ is a subring of a ring $A$ such that $S^o \in \gamma$, then also $(S^\star)^o \in \gamma$ where $S^\star$ denotes the ideal of $A$ generated by $S$.

**Proposition 2.** [5, Lemma 2.4] Let $\gamma$ be a radical. If $(\beta(A))^o \in \gamma$, then $\beta(A) \in \gamma$.

**Proposition 3.** [2, Corollary 1] If $\mathcal{M} \subseteq \mathcal{M}^o$ then $L(\mathcal{M}) \subseteq (L(\mathcal{M}))^o$ and $L_s(\mathcal{M}) \subseteq (L_s(\mathcal{M}))^o$.

**Proposition 4.** [4, Theorem 4] If a radical $\gamma$ is left strong and principally left hereditary, then $\gamma$ is normal.

**Proposition 5.** [2, Lemma 2] For any element $a$ of a ring $A$, $I = r(a)a$, where $r(a) = \{ x \in A \mid ax = 0 \}$ is an ideal of $Aa$ and $I^2 = 0$. In addition $Aa/I$ is a homomorphic image of $aA$.

**Proposition 6.** [5, Corollary 4.2] A radical $\gamma$ is hereditary and normal if and only if $\gamma$ is principally left strong, principally left hereditary and satisfies condition (H).

**Proposition 7.** [7, Theorem 6] A radical $\gamma$ is normal if and only if $\gamma$ is principally left or right hereditary and principally left or right strong.

**Proposition 8.** [6, Theorem 3.3] Let $\mathcal{M}$ be a homomorphically closed class of rings satisfying:

1) $\mathcal{M}$ contains all zero rings;
2) $\mathcal{M}$ is hereditary;
3) if $I \triangleleft A$, $I^2 = 0$ and $A/I \in \mathcal{M}$ then $A \in \mathcal{M}$.

Then $L_{ps}(\mathcal{M}) = \mathcal{M}_2$.

**Proposition 9.** [5, Theorem 5.1] The Behrens radical class $B$ is the largest principally left hereditary subclass of the Brown-McCoy radical class $\mathcal{G}$, in fact

$$B = \mathcal{MG},$$

where

$$\mathcal{MG} = \{ A \mid Aa \in \mathcal{G} \text{ for all } a \in A \}.$$

A ring $A$ is said to be (right) strongly prime if every non-zero ideal $I$ of $A$ contains a finite subset $F$ such that $r_A(F) = 0$, where $r_A(F) = \{ x \in A \mid Fx = 0 \}$.

The (right) strongly prime radical $S$ is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring $A$,

$$S(A) = \cap \{ I \triangleleft A \mid A/I \text{ is strongly prime} \}.$$ 

It is known that the radical $S$ is special: so, in particular, $S$ is hereditary and contains the prime radical $\beta$.

**Proposition 10.** [3, Corollary 1] The (right) strongly prime radical $S$ is right strong.
II.

**Proposition 11.** Let $\gamma$ be a principally left strong radical satisfying the conditions (H) and (Z). Then the largest hereditary subclass $\overline{\gamma}$ of $\gamma$ will be principally left strong.

**Proof.** Let $L \triangleleft A$ be such that $L \in \overline{\gamma}$ and $Lz \in \overline{\gamma}$ for every $z \in L$. Let $L^*$ be the ideal in $A$ generated by $L$, $L^* = L + LA$ and suppose $I \triangleleft L^*$. Then $IL \triangleleft L$, $IL \triangleleft I$ and $ILz \triangleleft Lz \in \overline{\gamma}$ for all $z \in L$. Since $\gamma$ satisfies condition (H), $\overline{\gamma}$ is hereditary, and so $ILz \in \gamma$ for all $z \in IL$. Since $\gamma$ is principally left strong $IL \subseteq \gamma(I)$. We have

$$I(L^*)^2 = I(L + LA)L^* = (IL + ILA)L^* \subseteq ILL^* \subseteq \gamma(I)L^* \subseteq \gamma(I).$$

So $I^3 \subseteq I(L^*)^2 \subseteq \gamma(I)$ and therefore $I/\gamma(I)$ is nilpotent, implying $I/\gamma(I) \in \beta$. We claim that $I^\circ \in \gamma$. Since $L \in \overline{\gamma} \subseteq \gamma$, by (Z) we conclude that $L^\circ \in \gamma$. Now Proposition 1 implies that $(L^*)^\circ \in \gamma$ and so by (H) it follows $I^\circ \in \gamma$. Hence $(I/\gamma(I))^\circ \in \gamma \cap \beta$ and applying Proposition 2 and taking into consideration that $I/\gamma(I)$ is nilpotent, we get

$$I/\beta(I) = \beta(I/\gamma(A)) \in \gamma.$$ 

Thus $I \in \gamma$ and so $\overline{\gamma}$ is principally left strong. \qed

**Corollary 12.** If a class $\mathcal{M}$ is hereditary and satisfies (Z) then $\mathcal{L}_{ps}(\mathcal{M})$ is hereditary.

**Proof.** By Proposition 3, we have $\mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^\circ$. Let $A \in \mathcal{L}_{ps}(\mathcal{M})$ then we get $A^\circ \in \mathcal{L}_s(\mathcal{M})$ and so $A^\circ \in \mathcal{L}(\mathcal{M})$. Since $\mathcal{L}(\mathcal{M})$ is hereditary, we conclude that $A^\circ \in \mathcal{L}(\mathcal{M})$ and so $A^\circ \in \mathcal{L}_{ps}(\mathcal{M})$. This means that $\mathcal{L}_{ps}(\mathcal{M})$ satisfies the conditions (Z) and (H). By Proposition 11, $\overline{\mathcal{L}_{ps}(\mathcal{M})}$ is principally left strong and $\mathcal{M} \subseteq \overline{\mathcal{L}_{ps}(\mathcal{M})} \subseteq \mathcal{L}_{ps}(\mathcal{M})$ and this implies $\overline{\mathcal{L}_{ps}(\mathcal{M})} = \mathcal{L}_{ps}(\mathcal{M})$. \qed

**Proposition 13.** Let $\gamma$ be a principally left strong radical satisfying the conditions (H) and (Z). Then $\gamma_r$ is left strong.

**Proof.** Let $L \triangleleft A$ and $L \in \gamma_r$ and let $K$ be a left ideal of $L^* = L + LA$. Since $L \in \gamma_r$, $kL \in \gamma$ for every $k \in K$. Let $R \triangleleft kL$. Then it is easy to see that $RkL \in \gamma$, and by conditions (Z) and (H), $R/kL \in \gamma$ and so $R \in \gamma$. Hence $kL \in \gamma_r$ for every $k \in K$. An argument similar to the proof of Proposition 5 will show that $(Lk + r(k)k)/r(k)k$ is a homomorphic image of $kL$, where $r(k) = \{x \in L^*/kx = 0\}$. Hence $(Lk + r(k)k)/r(k)k \in \gamma$. By (H) and (Z) we have $r(k)k \in \gamma$ and so $Lk \in \gamma$ for every $k \in K$. Therefore $Lk \subseteq \gamma(K)$ and $LK \subseteq \gamma(K)$. Clearly

$$K^3 \subseteq (L^*K)K \subseteq (LA^1K)K \subseteq LL^*K \subseteq LK \subseteq \gamma(K)$$

hence $K \in \gamma$ by Proposition 2. \qed

The next result is a generalization of [2, Corollary 4].

**Corollary 14.** If $\mathcal{M}$ is a right hereditary class with (Z), then $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary and $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$ (i.e. $\mathcal{L}_{ps}(\mathcal{M})$ is left and right hereditary).
**Proof.** By Corollary 12, \( \mathcal{L}_{ps}(M) \) satisfies condition (H). Let \( A \in \mathcal{L}_{ps}(M) \). Then it is easy to see that \( A^0 \in \mathcal{L}_{ps}(M) \). Hence \( \mathcal{L}_{ps}(M) \) satisfies condition (Z). Hence \( \mathcal{L}_{ps}(M)_r \) is a radical. By Proposition 13, \( \mathcal{L}_{ps}(M)_r \) is left strong. Since \( M \subseteq \mathcal{L}_{ps}(M)_r \) we get \( M \subseteq \mathcal{L}_{ps}(M) \subseteq \mathcal{L}_{ps}(M) \subseteq L_s(M) \) and \( \mathcal{L}_{ps}(M)_r = L_s(M) \). Hence \( \mathcal{L}_{ps}(M) = L_s(M) \). Since \( \mathcal{L}_{ps}(M)_r \) is right hereditary and left strong, we have that \( \mathcal{L}_{ps}(M) \) is one-sided hereditary.

**Theorem 15.** Let \( \gamma \neq 0 \) be a principally left strong radical with (Z) and (H). Then \( \gamma_r \) is contained in \( \gamma \) as a largest nonzero hereditary and normal subradical. Furthermore, \( \gamma \) is contained in \( \gamma \) as a largest non-zero hereditary principally left strong subradical.

**Proof.** Let \( 0 \neq A \in \gamma \). By (Z), \( A^0 \in \gamma \) and by (H), \( A^0 \in \gamma_r \). All zero-rings of \( \gamma \) are in \( \gamma_r \) and so \( \gamma_r \neq 0 \). Hence \( \gamma_r \) satisfies conditions (Z) and (H). By Propositions 13, 6 and 4, \( \gamma \) is normal and hereditary.

The second part of the theorem follows from Proposition 11.

**Corollary 16.** The largest left hereditary subclass \( S_1 \) of strongly prime radical \( S \) is the largest normal radical contained in \( S \).

**Theorem 17.** The following statements are equivalent for a radical \( \gamma \).

1) \( \gamma \) is hereditary and normal.

2) \( \gamma \) is left or right principally hereditary, principally left or right strong and satisfies condition (H).

3) There exists a principally left (right, respectively) strong radical \( \delta \) such that \( \delta_r = \gamma \) (\( \delta_l = \gamma \), respectively) and satisfies conditions (Z) and (H).

4) There exists a right (left, respectively) hereditary class \( M \) of rings satisfying (Z) such that \( \gamma = \mathcal{L}_{ps}(M) \) (\( \gamma = \mathcal{L}_{ps}(M)_r \), respectively), where \( \mathcal{L}_{ps}(M) \) is principally right strong generated by \( M \).

**Proof.** 2) \( \implies \) 1): By Proposition 7, \( \gamma \) is normal and by Proposition 6, \( \gamma \) is hereditary.

1) \( \implies \) 3): We claim that \( \gamma \) is one-sided hereditary. So let \( L \cap A \in \gamma \). Since \( \gamma \) is normal, \( \gamma \) is principally left hereditary, so \( Aa \in \gamma \), for all \( a \in L \). Therefore \( Aa \cdot z \in \gamma \) for every \( z \in Aa \). Hence \( Aa \subseteq \gamma(L) \) for all \( a \in L \), and this gives \( L^2 \subseteq \gamma(L) \). Again, since \( \gamma \) is normal and satisfies condition (Z), \( A^0 \in \gamma \) and by hereditariness \( L^0 \in \gamma \). Therefore \( L \in \gamma \). Right hereditariness is proved analogously. Now we choose \( \delta \) to be \( \gamma \), \( \delta = \gamma \) and we have \( \gamma = \delta = \delta_l = \delta_r \).

3) \( \implies \) 4): We choose \( M = \delta^0 \) (\( M = \delta_l \), respectively). Then \( \delta_r = \mathcal{L}_{ps}(\delta_l) = \mathcal{L}_{ps}(M) \) (\( \delta_l = \mathcal{L}_{ps}(\delta_l) = \mathcal{L}_{ps}(M)_r \), respectively) by Proposition 13 and clearly \( \delta_r \) satisfies (Z).

4) \( \implies \) 2): By Corollary 14, \( \gamma = \mathcal{L}_{ps}(M) \) (\( \gamma = \mathcal{L}_{ps}(M)_r \)) is one-sided hereditary and left strong. Hence by Proposition 4 it is normal. It is easy to see that \( \gamma \) satisfies 2).

**Proposition 18.** Let \( \gamma \) be a supernilpotent radical and let us assume that \( \gamma_l = \gamma_r \) is the largest principally left hereditary subclass of \( \gamma \) which we will denote by \( \delta \). Then

\[
\mathcal{L}_{ps}(\gamma) = \mathcal{L}_{ps}(\delta) \lor \gamma
\]

where \( \lor \) denotes the union in the lattice of all radicals (i.e. the lower radical determined by the union of the components).
Proof. Clearly $\mathcal{L}_{ps}(\delta) \cup \gamma \subseteq \mathcal{L}_{ps}(\gamma)$. Conversely, let $A \in \mathcal{L}_{ps}(\gamma)$. Under our hypothesis, we can apply Proposition 8 and so $\mathcal{L}_{ps}(\gamma) = \gamma_2$. Thus any non-zero homomorphic image $A'$ of $A$ has a non-zero $\gamma$-ideal or a non-zero left ideal $L$ such that $La \in \gamma$ for all $a \in L \cup \{1\}$. Using our hypothesis again, we conclude that $L \in \delta$ and therefore the $\mathcal{L}_{ps}(\delta)$-radical of $A'$ is nonzero. Hence $A'$ has a nonzero ideal in $\mathcal{L}_{ps}(\delta) \cup \gamma$ and so $A \in \mathcal{L}_{ps}(\delta) \cup \gamma$. □

Corollary 19. $\mathcal{L}_{ps}(\mathcal{G}) = \mathcal{L}_{ps}(\mathcal{B}) \cup \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{B}_2 \cup \mathcal{G}$.

Proof. By Proposition 9, the Brown-McCoy radical satisfies the assumption of Proposition 18, in fact, $\mathcal{MG} = \mathcal{Gl} = \mathcal{Gr} = \mathcal{B}$.

Remark. This corollary can also be obtained as an application of Proposition 8 to the radicals $\mathcal{G}$ and $\mathcal{B}$.

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