Hereditariness, Strongness and Relationship between Brown-McCoy and Behrens Radicals

S. Tumurbat    H. Zand

Department of Algebra, University of Mongolia
P.O. Box 75, Ulaan Baatar 20, Mongolia
e-mail: tumur@www.com

Open University, Milton Keynes
MK7 6AA, England
e-mail: h.zand@open.ac.uk

Abstract. In this paper we explore the properties of being hereditary and being strong among the radicals of associative rings, and prove certain results such as a relationship between Brown-McCoy and Behrens radicals.

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I.

In this paper rings are all associative, but not necessarily with a unit element. As usual, $I \triangleleft A$ and $L \triangleleft_l A$ ($R \triangleleft_r A$) denote that $I$ is an ideal and $L$ is a left ideal ($R$ is a right ideal) in $A$, respectively. $A^o$ will stand for the ring on the additive group $(A, +)$ with multiplication $xy = 0$, for all $x, y \in A$.

Let us recall that a (Kurosh-Amitsur) radical $\gamma$ is a class of rings which is closed under homomorphisms, extensions ($I$ and $A/I$ in $\gamma$ imply $A$ in $\gamma$), and has the inductive property (if $I_1 \subseteq \cdots \subseteq I_\lambda \subseteq \cdots$ is a chain of ideals, $A = \cup I_\lambda$, and each $I_\lambda$ is in $\gamma$, then $A$ is in $\gamma$).

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The unique largest $\gamma$-ideal $\gamma(A)$ of $A$ is then the $\gamma$-radical of $A$. A hereditary radical containing all nilpotent rings is called a supernilpotent radical. Let $\mathcal{M}$ be a class of rings. Put

$$\overline{\mathcal{M}} = \{ A \mid \text{every ideal of } A \text{ is in } \mathcal{M} \}.$$ 

A radical $\gamma$ is said to be principally left (right) hereditary if $a \in A \in \gamma$ implies $Aa \in \gamma$ ($aA \in \gamma$, respectively). A radical $\gamma$ is said to be left (right) strong if $L \lhd_A (R \lhd_A)$ and $L \in \gamma$ ($R \in \gamma$, respectively). A radical $\gamma$ is normal if $\gamma$ is left strong and principally left hereditary. We shall make use of the following condition a left ideal $L$ of a ring $A$ may satisfy with respect to a class $\mathcal{M}$ of rings:

$$(*) \; L \lhd_A \text{ and } Lz \in \mathcal{M} \text{ for all } z \in L \cup \{1\}.$$ 

A radical $\gamma$ is said to be principally left strongly if $L \subseteq \gamma(A)$ whenever the left ideal $L$ of a ring $A$ satisfies condition $(\ast)$ with respect to the class $\gamma(= \mathcal{M})$. Principally right strongness is defined analogously.

We will focus on two conditions that a class $\mathcal{M}$ can satisfy.

(H) If $A^o \in \mathcal{M}$ then $S \in \mathcal{M}$ for every subring $S \subseteq A^o$.

(Z) If $A \in \mathcal{M}$ then $A^o \in \mathcal{M}$.

A class $\mathcal{M}$ of rings is said to be regular if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the upper radical operator $U$ yields a radical class

$$UM = \{ A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M} \}.$$ 

Recall that the Baer radical $\beta$ is the upper radical determined by all prime rings, the Brown-McCoy radical $G$ is the upper radical determined by all simple rings with unity element, and the Behrens radical $B$ is the upper radical of all subdirectly irreducible rings having a nonzero idempotent in their hearts.

The lower principally left strongly radical construction $L_{ps}(\mathcal{M})$ is similar to the lower (left) strong radical construction $L_s(\mathcal{M})$ (see [1]).

We shall construct the lower principally left strongly radical (see also [7]) in the following way. Let $\mathcal{M}$ be a homomorphically closed class of rings and define $\mathcal{M} = \mathcal{M}_1$,

$$\mathcal{M}_{\alpha+1} = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a } \text{nonzero left ideal with } \ast \text{ in } \mathcal{M}_{\alpha} \text{ or a nonzero } \text{ideal } I \in \mathcal{M}_{\alpha} \end{array} \right\}$$

for ordinals $\alpha \geq 1$ and $\mathcal{M}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{M}_\alpha$ for limit ordinals $\lambda$. In particular,

$$\mathcal{M}_2 = \left\{ A \mid \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a nonzero left ideal with } \ast \text{ in } \mathcal{M} \text{ or a nonzero ideal } \text{ideal } I \in \mathcal{M} \end{array} \right\}.$$ 

The class $L_{ps}(\mathcal{M}) = \bigcup_{\alpha} \mathcal{M}_\alpha$ is called the lower principally left strongly radical class. As shown in [6] $L_{ps}(\mathcal{M})$ is the smallest principally left strongly radical containing $\mathcal{M}$ and

$$\mathcal{M} \subseteq L(\mathcal{M}) \subseteq L_{ps}(\mathcal{M}) \subseteq L_s(\mathcal{M}).$$
For any class $\mathcal{M}$ let us define $\mathcal{M}^o = \{ A \mid A^o \in \mathcal{M} \}$. It is easy to see that if $\mathcal{M}$ is a radical then so is $\mathcal{M}^o$. Let 
\[ \gamma_l = \{ A \in \gamma \mid \text{every left ideal of } A \text{ is in } \gamma \} \]
and 
\[ \gamma_r = \{ A \in \gamma \mid \text{every right ideal of } A \text{ is in } \gamma \} \].

Next, we recall some results which will be used later on.

**Proposition 1.** [2, Lemma 1] Let $\gamma$ be a radical. If $S$ is a subring of a ring $A$ such that $S^o \in \gamma$, then also $(S^*)^o \in \gamma$ where $S^*$ denotes the ideal of $A$ generated by $S$.

**Proposition 2.** [5, Lemma 2.4] Let $\gamma$ be a radical. If $(\beta(A))^o \in \gamma$, then $\beta(A) \in \gamma$.

**Proposition 3.** [2, Corollary 1] If $\mathcal{M} \subseteq \mathcal{M}^o$ then $\mathcal{L}(\mathcal{M}) \subseteq (\mathcal{L}(\mathcal{M}))^o$ and $\mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^o$.

**Proposition 4.** [4, Theorem 4] If a radical $\gamma$ is left strong and principally left hereditary, then $\gamma$ is normal.

**Proposition 5.** [2, Lemma 2] For any element $a$ of a ring $A$, $I = r(a)a$, where $r(a) = \{ x \in A \mid ax = 0 \}$ is an ideal of $Aa$ and $I^2 = 0$. In addition $Aa/I$ is a homomorphoic image of $aA$.

**Proposition 6.** [5, Corollary 4.2] A radical $\gamma$ is hereditary and normal if and only if $\gamma$ is principally left strong, principally left hereditary and satisfies condition (H).

**Proposition 7.** [7, Theorem 6] A radical $\gamma$ is normal if and only if $\gamma$ is principally left or right hereditary and principally left or right strong.

**Proposition 8.** [6, Theorem 3.3] Let $\mathcal{M}$ be a homomorphically closed class of rings satisfying:

1) $\mathcal{M}$ contains all zero rings;
2) $\mathcal{M}$ is hereditary;
3) if $I \triangleleft A$, $I^2 = 0$ and $A/I \in \mathcal{M}$ then $A \in \mathcal{M}$.

Then $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{M}_2$.

**Proposition 9.** [5, Theorem 5.1] The Behrens radical class $\mathcal{B}$ is the largest principally left hereditary subclass of the Brown-McCoy radical class $\mathcal{G}$, in fact 
\[ \mathcal{B} = \mathcal{M}\mathcal{G}, \]
where 
\[ \mathcal{M}\mathcal{G} = \{ A \mid Aa \in \mathcal{G} \text{ for all } a \in A \}. \]

A ring $A$ is said to be (right) strongly prime if every non-zero ideal $I$ of $A$ contains a finite subset $F$ such that $r_A(F) = 0$, where $r_A(F) = \{ x \in A \mid Fx = 0 \}$.

The (right) strongly prime radical $S$ is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring $A$, 
\[ S(A) = \cap \{ I \triangleleft A \mid A/I \text{ is strongly prime} \}. \]

It is known that the radical $S$ is special: so, in particular, $S$ is hereditary and contains the prime radical $\beta$.

**Proposition 10.** [3, Corollary 1] The (right) strongly prime radical $S$ is right strong.
II.

**Proposition 11.** Let $\gamma$ be a principally left strong radical satisfying the conditions (H) and (Z). Then the largest hereditary subclass $\overline{\gamma}$ of $\gamma$ will be principally left strong.

*Proof.* Let $L \triangleleft A$ be such that $L \in \overline{\gamma}$ and $Lz \in \overline{\gamma}$ for every $z \in L$. Let $L^*$ be the ideal in $A$ generated by $L$, $L^* = L + LA$ and suppose $I \triangleleft L^*$. Then $IL \triangleleft L$, $IL \triangleleft I$ and $ILz \triangleleft Lz \in \overline{\gamma}$ for all $z \in L$. Since $\gamma$ satisfies condition (H), $\overline{\gamma}$ is hereditary, and so $ILz \in \overline{\gamma}$ for all $z \in IL$. Since $\gamma$ is principally left strong $IL \subseteq \gamma(I)$. We have

$$I(L^*)^2 = I(L + LA)L^* = (IL + ILA)L^* \subseteq ILL^* \subseteq \gamma(I)L^* \subseteq \gamma(I).$$

So $I^3 \subseteq I(L^*)^2 \subseteq \gamma(I)$ and therefore $I/\gamma(I)$ is nilpotent, implying $I/\gamma(I) \in \beta$. We claim that $I^o \in \gamma$. Since $L \in \overline{\gamma} \subseteq \gamma$, by (Z) we conclude that $L^o \in \gamma$. Now Proposition 1 implies that $(L^*)^o \in \gamma$ and so by (H) it follows $I^o \in \gamma$. Hence $(I/\gamma(I))^o \in \gamma \cap \beta$ and applying Proposition 2 and taking into consideration that $I/\gamma(I)$ is nilpotent, we get

$$I/\beta(I) = \beta(I/\gamma(A)) \in \gamma.$$

Thus $I \in \gamma$ and so $\overline{\gamma}$ is principally left strong. \hfill \qed

**Corollary 12.** If a class $\mathcal{M}$ is hereditary and satisfies (Z) then $\mathcal{L}_{ps}(\mathcal{M})$ is hereditary.

*Proof.* By Proposition 3, we have $\mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^o$. Let $A \in \mathcal{L}_{ps}(\mathcal{M})$ then we get $A^o \in \mathcal{L}_s(\mathcal{M})$ and so $A^o \in \mathcal{L}(\mathcal{M})$. Since $\mathcal{L}(\mathcal{M})$ is hereditary, we conclude that $A^o \in \mathcal{L}(\mathcal{M})$ and so $A^o \in \mathcal{L}_{ps}(\mathcal{M})$. This means that $\mathcal{L}_{ps}(\mathcal{M})$ satisfies the conditions (Z) and (H). By Proposition 11, $\mathcal{L}_{ps}(\mathcal{M})$ is principally left strong and $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_{ps}(\mathcal{M})$ and this implies $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_{ps}(\mathcal{M})$. \hfill \qed

**Proposition 13.** Let $\gamma$ be a principally left strong radical satisfying the conditions (H) and (Z). Then $\gamma_r$ is left strong.

*Proof.* Let $L \triangleleft A$ and $L \in \gamma_r$ and let $K$ be a left ideal of $L^* = L + LA$. Since $L \in \gamma_r$, $kL \in \gamma$ for every $k \in K$. Let $R \triangleleft kL$. Then it is easy to see that $RkL \in \gamma$, and by conditions (Z) and (H), $R/kL \in \gamma$ and so $R \in \gamma$. Hence $kL \in \gamma_r$ for every $k \in K$. An argument similar to the proof of Proposition 5 will show that $(Lk + r(k)k)/r(k)k$ is a homomorphic image of $kL$, where $r(k) = \{x \in L^*/kx = 0\}$. Hence $(Lk + r(k)k)/r(k)k \in \gamma$. By (H) and (Z) we have $r(k)k \in \gamma$ and so $Lk \in \gamma$ for every $k \in K$. Therefore $Lk \subseteq \gamma(K)$ and $LK \subseteq \gamma(K)$. Clearly

$$K^3 \subseteq (L^*K)K \subseteq (LA^1K)K \subseteq LL^*K \subseteq LK \subseteq \gamma(K)$$

hence $K \in \gamma$ by Proposition 2. \hfill \qed

The next result is a generalization of [2, Corollary 4].

**Corollary 14.** If $\mathcal{M}$ is a right hereditary class with (Z), then $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary and $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$ (i.e. $\mathcal{L}_{ps}(\mathcal{M})$ is left and right hereditary).
Proof. By Corollary 12, $\mathcal{L}_{ps}(\mathcal{M})$ satisfies condition (H). Let $A \in \mathcal{L}_{ps}(\mathcal{M})$. Then it is easy to see that $A^o \in \mathcal{L}_{ps}(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M})$, $\mathcal{M}$ satisfies condition (Z). Hence $\mathcal{L}_{ps}(\mathcal{M})_r$ is a radical. By Proposition 13, $\mathcal{L}_{ps}(\mathcal{M})_r$ is left strong. Since $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r$, we get $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_{s}(\mathcal{M})$ and $\mathcal{L}_{ps}(\mathcal{M})_r = \mathcal{L}_{s}(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_{s}(\mathcal{M})$. Since $\mathcal{L}_{ps}(\mathcal{M})_r$ is right hereditary and left strong, we have that $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary. \hfill \Box

Theorem 15. Let $\gamma \neq 0$ be a principally left strong radical with (Z) and (H). Then $\gamma_r$ is contained in $\gamma$ as a largest non-zero hereditary and normal subradical. Furthermore, $\gamma$ is contained in $\gamma$ as a largest non-zero hereditary principally left strong subradical.

Proof. Let $0 \neq A \in \gamma$. By (Z), $A^o \in \gamma$ and by (H), $A^o \in \gamma_r$. All zero-rings of $\gamma$ are in $\gamma_r$ and so $\gamma_r \neq 0$. Hence $\gamma_r$ satisfies conditions (Z) and (H). By Propositions 13, 6 and 4, $\gamma$ is normal and hereditary.

The second part of the theorem follows from Proposition 11. \hfill \Box

Corollary 16. The largest left hereditary subclass $S_1$ of strongly prime radical $S$ is the largest normal radical contained in $S$.

Theorem 17. The following statements are equivalent for a radical $\gamma$.

1) $\gamma$ is hereditary and normal.
2) $\gamma$ is left or right principally hereditary, principally left or right strong and satisfies condition (H).
3) There exists a principally left (right, respectively) strong radical $\delta$ such that $\delta_r = \gamma$ ($\delta_l = \gamma$, respectively) and satisfies conditions (Z) and (H).
4) There exists a right (left, respectively) hereditary class $\mathcal{M}$ of rings satisfying (Z) such that $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}'_{ps}(\mathcal{M})$, respectively), where $\mathcal{L}'_{ps}(\mathcal{M})$ is principally right strong radical generated by $\mathcal{M}$.

Proof. 2) $\implies$ 1): By Proposition 7, $\gamma$ is normal and by Proposition 6, $\gamma$ is hereditary.

1) $\implies$ 3): We claim that $\gamma$ is one-sided hereditary. So let $L \trianglelefteq A \in \gamma$. Since $\gamma$ is normal, $\gamma$ is principally left hereditary, so $Aa \in \gamma$, for all $a \in L$. Therefore $Aa \cdot z \in \gamma$ for every $z \in Aa$. Hence $Aa \subseteq \gamma(L)$ for all $a \in L$, and this gives $L^2 \subseteq \gamma(L)$. Again, since $\gamma$ is normal and satisfies condition (Z), $A^o \in \gamma$ and by hereditariness $L^o \in \gamma$. Therefore $L \in \gamma$. Right hereditariness is proved analogously. Now we choose $\delta$ to be $\gamma$, $\delta = \gamma$ and we have $\gamma = \delta = \delta_l = \delta_r$.

3) $\implies$ 4): We choose $\mathcal{M} = \delta^o$ ($\mathcal{M} = \delta_l$, respectively). Then $\delta_r = \mathcal{L}_{ps}(\delta_l) = \mathcal{L}_{ps}(\mathcal{M})$ ($\delta_l = \mathcal{L}'_{ps}(\delta_l) = \mathcal{L}'_{ps}(\mathcal{M})$, respectively) by Proposition 13 and clearly $\delta_r$ satisfies (Z).

4) $\implies$ 2): By Corollary 14, $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}'_{ps}(\mathcal{M})$) is one-sided hereditary and left strong. Hence by Proposition 4 it is normal. It is easy to see that $\gamma$ satisfies 2). \hfill \Box

Proposition 18. Let $\gamma$ be a supernilpotent radical and let us assume that $\gamma_l = \gamma_r$ is the largest principally left hereditary subclass of $\gamma$ which we will denote by $\delta$. Then

$$\mathcal{L}_{ps}(\gamma) = \mathcal{L}_{ps}(\delta) \lor \gamma$$

where $\lor$ denotes the union in the lattice of all radicals (i.e. the lower radical determined by the union of the components).
Proof. Clearly $L_{ps}(\delta) \cup \gamma \subseteq L_{ps}(\gamma)$. Conversely, let $A \in L_{ps}(\gamma)$. Under our hypothesis, we can apply Proposition 8 and so $L_{ps}(\gamma) = \gamma_2$. Thus any non-zero homomorphic image $A'$ of $A$ has a non-zero $\gamma$-ideal or a nonzero left ideal $L$ such that $La \in \gamma$ for all $a \in L \cup \{1\}$. Using our hypothesis again, we conclude that $L \in \delta$ and therefore the $L_{ps}(\delta)$-radical of $A'$ is nonzero. Hence $A'$ has a nonzero ideal in $L_{ps}(\delta) \cup \gamma$ and so $A \in L_{ps}(\delta) \cup \gamma$.

\begin{corollary}
$L_{ps}(G) = L_{ps}(B) \cup G$ and $G_2 = B_2 \cup G$.
\end{corollary}

Proof. By Proposition 9, the Brown-McCoy radical satisfies the assumption of Proposition 18, in fact, $MG = G_l = G_r = B$.

Remark. This corollary can also be obtained as an application of Proposition 8 to the radicals $G$ and $B$.

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References


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